OSCILLATION CRITERIA FOR FIRST-ORDER DELAY EQUATIONS

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ABSTRACT

This paper is concerned with the oscillatory behavior of first-order delay differential equations of the form

$$x'(t) + p(t)x(\tau(t)) = 0, \qquad t \geq t_0, \tag{1}$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+), \mathbb{R}^+ = [0, \infty), \tau(t)$ is non-decreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$. Let the numbers k and L be defined by

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \text{ and } L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds.$$

It is proved here that when $L < 1$ and $0 < k \leq \frac{1}{e}$ all solutions of Eq. (1) oscillate in several cases in which the condition

$$L > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2k\lambda_1}}{\lambda_1}$$

holds, where λ_1 is the smaller root of the equation $\lambda = e^{k\lambda}$.

[0] *Key Words:* Oscillation; delay differential equations.
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1 Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions of the differential equation

$$x'(t) + p(t)\, x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where the functions $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$(here $\mathbb{R}^+ = [0, \infty)$), $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, has been the subject of many investigations. See, for example, [1-27] and the references cited therein.

By a solution of Eq. (1) we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (1) is satisfied for $t \geq T_0$. Such a solution is called oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory.

The first systematic study for the oscillation of all solutions of Eq. (1) was made by Myshkis. In 1950 [24] he proved that every solution of Eq. (1) oscillates if

$$\limsup_{t\to\infty}[t - \tau(t)] < \infty, \quad \liminf_{t\to\infty}[t - \tau(t)] \cdot \liminf_{t\to\infty} p(t) > \frac{1}{e}. \tag{C_1}$$

In 1972, Ladas, Lakshmikantham and Papadakis [19] proved that the same conclusion holds if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1. \tag{C_2}$$

In 1979 Ladas [18] and in 1982 Koplatadze and Chanturija [14] improved (C_1) to

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}. \tag{C_3}$$

Concerning the constant $\frac{1}{e}$ in (C_3), it is to be pointed out that if the inequality

$$\int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e}$$

holds eventually, then, according to a result in [14], (1) has a non-oscillatory solution.

In 1982 Ladas, Sficas and Stavroulakis [20] and in 1984 Fukagai and Kusano [10] established oscillation criteria (of the type of conditions $(C_2)$and (C_3)) for Eq. (1) with *oscillating* coefficient $p(t)$.

It is obvious that there is a gap between the conditions (C_2) and (C_3) when the limit

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$$

does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [9] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible nonoscillatory solutions $x(t)$ of Eq. (1). Their result, when formulated in terms of the numbers k and L defined by

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \quad \text{and} \quad L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds,$$

says that all the solutions of Eq (1) are oscillatory, if $0 < k \le \frac{1}{e}$ and

$$L > 1 - \frac{k^2}{4}. \tag{C_4}$$

Since then several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$. In 1991 Jian Chao [2] derived the condition

$$L > 1 - \frac{k^2}{2(1-k)}, \tag{C_5}$$

while in 1992 Yu and Wang [26] and Yu, Wang, Zhang and Qian [27] obtained the condition

$$L > 1 - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}. \tag{C_6}$$

In 1990 Elbert and Stavroulakis [7] and in 1991 Kwong [17], using different techniques, improved (C_4), in the case where $0 < k \le \frac{1}{e}$, to the conditions

$$L > 1 - (1 - \frac{1}{\sqrt{\lambda_1}})^2 \tag{C_7}$$

and

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1}, \tag{C_8}$$

respectively, where $\lambda_1$is the smaller root of the equation

$$\lambda = e^{\kappa\lambda}. \tag{2}$$

In 1994 Koplatadze and Kvinikadze [14] improved (C_6), while in 1998 Philos and Sficas [25] , in 1999 Jaroš and Stavroulakis [11] and in 2000 Kon, Sficas and Stavroulakis [12] derived the conditions

$$L > 1 - \frac{k^2}{2(1-k)} - \frac{k^2}{2}\lambda_1, \tag{C_9}$$

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2} \tag{C_{10}}$$

and

$$L > 2k + \frac{2}{\lambda_1} - 1, \qquad (C_{11})$$

respectively, where λ_1 is the smaller root of Eq. (2).

Following this historical (and chronological) review we also mention that in the case where

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}$$

this problem has been studied in 1993 by Elbert and Stavroulakis [8] and in 1995 by Kozakiewicz [16], Li [22], [23] and by Domshlak and Stavroulakis [5].

The purpose of this paper is to improve the methods previously used to show that the conditions (C_2) and (C_4)-(C_{11}) may be weakened to

$$L > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2k\lambda_1}}{\lambda_1}, \qquad (C_{12})$$

where λ_1 is the smaller root of the equation $\lambda = e^{\kappa\lambda}$.

It is to be noted that as $k \to 0$, then all conditions (C_4)-(C_{11}) reduce to the condition (C_2), i.e.

$$L > 1.$$

However our condition (C_{12}) leads to

$$L > \sqrt{3} - 1 \approx 0.732$$

which is an essential improvement. Moreover (C_{12}) improves all the above conditions when $0 < k \leq \frac{1}{e}$ as well. For illustrative purpose, we give the values of the lower bound on L under these conditions when $k = \frac{1}{e}$:

(C_2):	1.000000000
(C_4):	0.966166179
(C_5):	0.892951367
(C_6):	0.863457014
(C_7):	0.845181878
(C_8):	0.735758882
(C_9):	0.709011646
(C_{10}):	0.599215896
(C_{11}):	0.471517764
(C_{12}):	0.459987065

We see that our condition (C_{12}) essentially improves all the known results in the literature.

2 Main Results

In what follows we will denote by k and L the lower and upper limits of the average $\int_{\tau(t)}^{t} p(s)ds$ as $t \to \infty$, respectively, i.e.

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$$

and

$$L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds.$$

Set

$$w(t) = \frac{x(\tau(t))}{x(t)}.$$

We begin with the preliminary analysis of asymptotic behavior of the function $w(t)$ for a possible nonoscillatory solution $x(t)$ of Eq. (1) in the case that $k \leq \frac{1}{e}$. For this purpose, assume that (1) has a solution $x(t)$ which is positive for all large t. Dividing first Eq. (1) by $x(t)$ and then integrating it from $\tau(t)$ to t leads to the integral equality

$$w(t) = \exp \int_{\tau(t)}^{t} p(s)w(s)ds \tag{3}$$

which holds for all sufficiently large t.

For the next lemmata see [12].

Lemma 1 *Suppose that $k > 0$ and Eq. (1) has an eventually positive solution $x(t)$. Then $k \leq 1/e$ and*

$$\lambda_1 \leq \liminf_{t\to\infty} w(t) \leq \lambda_2,$$

where λ_1 is the smaller and λ_2 the greater root of the equation $\lambda = e^{\kappa\lambda}$.

Lemma 2 *Let $0 < k \leq 1/e$ and let $x(t)$ be an eventually positive solution of Eq. (1). Assume that $\tau(t)$ is continuously differentiable and that there exists $\omega > 0$ such that*

$$p(\tau(t))\tau'(t) \geq \omega p(t) \tag{4}$$

eventually for all t. Then

$$\limsup_{t\to\infty} w(t) \leq \frac{2}{1 - k - \sqrt{(1-k)^2 - 4A}},$$

where A is given by

$$A = \frac{e^{\lambda_1 \omega k} - \lambda_1 \omega k - 1}{(\lambda_1 \omega)^2} \tag{5}$$

Remark 1 *It is easy to see that (4) implies that*

$$\int_{\tau(u)}^{\tau(t)} p(s)ds \geq \omega \int_u^t p(s)ds \quad \text{for all } \tau(t) \leq u \leq t. \tag{4'}$$

Indeed, the function

$$v(u) = \int_{\tau(u)}^{\tau(t)} p(s)ds - \omega \int_u^t p(s)ds, \qquad \tau(t) \leq u \leq t,$$

satisfies the condition

$$v(t) = 0,$$

and

$$v'(u) = -p(\tau(u))\tau'(u) + \omega p(u) \leq 0.$$

If $p(t) > 0$ *eventually for all* t *and*

$$\liminf_{t\to\infty} \frac{p(\tau(t))\tau'(t)}{p(t)} = \omega_0 > 0,$$

then ω *can be any number satisfying* $0 < \omega < \omega_0$. *Besides the case* $p(t) \equiv p > 0$, $\tau(t) = t - \tau$ *or the case* $\tau(t) = t - \tau$ *and* $p(t)$ *is* $\tau-$*periodic, there exists a class of functions which satisfy* (4). *Such a function is given in the Example below.*

Lemma 3 *Let* $0 < k \leq \frac{1}{e}$ *and let* $x(t)$ *be an eventually positive solution of Eq.(1). Assume that condition (4) is satisfied. Then*

$$L \leq \frac{\ln \lambda_1}{\lambda_1} + \frac{-1 + \sqrt{1 + 2\omega - 2\omega\lambda_1 M}}{\omega\lambda_1}, \tag{6}$$

where λ_1 *is the smaller root of the equation* $\lambda = e^{k\lambda}$ *and* $M = \liminf\limits_{t\to\infty} \frac{x(t)}{x(\tau(t))}$.

Proof. Let θ be any number in $(1/\lambda_1, 1)$. From Lemma 1 and the definition of M, there is a $T_1 > T$ such that

$$\frac{x(\tau(t))}{x(t)} > \theta\lambda_1, \quad t \geq T_1, \tag{7}$$

and

$$\frac{x(t)}{x(\tau(t))} > \theta M, \quad t \geq T_1. \tag{8}$$

Now let $t \geq T_1$. Since the function $g(s) = x(\tau(t))/x(s)$ is continuous, $g(\tau(t)) = 1 < \theta\lambda_1$, and $g(t) > \theta\lambda_1$, there is $t^* \equiv t^*(t) \in (\tau(t), t)$ such that

$$\frac{x(\tau(t))}{x(t^*)} = \theta\lambda_1. \tag{9}$$

Dividing (1) by $x(t)$, integrating from $\tau(t)$ to t^* and taking into account (7), yields

$$\int_{\tau(t)}^{t^*} p(s)ds \leq -\frac{1}{\theta\lambda_1}\int_{\tau(t)}^{t^*}\frac{x'(s)}{x(s)}ds = \frac{\ln(\theta\lambda_1)}{\theta\lambda_1} \tag{10}$$

Next we try to find an analogous inequality for

$$\Lambda := \int_{t^*}^{t} p(s)ds.$$

Integrating (1) from $\tau(s)$ to $\tau(t)$, we have

$$x(\tau(s)) - x(\tau(t)) = \int_{\tau(s)}^{\tau(t)} p(u)x(\tau(u))du, \qquad t^* \leq s \leq t.$$

Thus, integrating (1) from t^* to t and using (4′), we obtain

$$x(t^*) - x(t) = \int_{t^*}^{t} p(s)x(\tau(s))ds =$$

$$= \int_{t^*}^{t} p(s)[x(\tau(t)) + \int_{\tau(s)}^{\tau(t)} p(u)x(\tau(u))du]ds$$

$$\geq x(\tau(t))\int_{t^*}^{t} p(s)ds + x(\tau^2(t))[\int_{t^*}^{t} p(s)(\int_{\tau(s)}^{\tau(t)} p(u)du)ds]$$

$$\geq x(\tau(t))\int_{t^*}^{t} p(s)ds + \omega x(\tau^2(t))\int_{t^*}^{t} p(s)(\int_{s}^{t} p(u)du)ds$$

$$= x(\tau(t))\int_{t^*}^{t} p(s)ds + \frac{\omega}{2}x(\tau^2(t))(\int_{t^*}^{t} p(s)ds)^2$$

$$= \Lambda x(\tau(t)) + \frac{\omega}{2}\Lambda^2 x(\tau^2(t)),$$

where $\tau^2(t) \equiv \tau(\tau(t))$. Therefore

$$\Lambda + \frac{\Lambda^2}{2}\omega\frac{x(\tau^2(t))}{x(\tau(t))} \leq \frac{x(t^*)}{x(\tau(t))} - \frac{x(t)}{x(\tau(t))}$$

(and taking into account (9) and (8))

$$\leq \frac{1}{\theta\lambda_1} - \theta M.$$

Since, by (7),

$$\frac{x(\tau^2(t))}{x(\tau(t))} > \theta\lambda_1,$$

we obtain

$$\Lambda + \frac{\Lambda^2}{2}\omega\theta\lambda_1 \le \frac{1}{\theta\lambda_1} - \theta M$$

or

$$\Lambda^2\frac{\omega\theta\lambda_1}{2} + \Lambda + (\theta M - \frac{1}{\theta\lambda_1}) \le 0,$$

which leads to

$$\Lambda \le \frac{-1+\sqrt{1-2\omega\theta\lambda_1(\theta M - \frac{1}{\theta\lambda_1})}}{\omega\theta\lambda_1} = \frac{-1+\sqrt{1+2\omega-2\omega\theta^2\lambda_1 M}}{\omega\theta\lambda_1},$$

since the other root is negative. Adding (10) and the last inequality, we obtain

$$\int_{\tau(t)}^{t} p(s)ds \le \frac{\ln(\theta\lambda_1)}{\theta\lambda_1} + \frac{-1+\sqrt{1+2\omega-2\omega\theta^2\lambda_1 M}}{\omega\theta\lambda_1}.$$

Letting $\theta \to 1$ completes the proof. ■

Theorem. *Consider the differential equation (1) and let $L < 1$, $0 < k \le \frac{1}{e}$ and there exists $\omega > 0$ such that (4) be satisfied. Assume that*

$$L > \frac{\ln\lambda_1}{\lambda_1} + \frac{-1+\sqrt{1+2\omega-2\omega\lambda_1 B}}{\omega\lambda_1}, \tag{11}$$

where λ_1 is the smaller root of the equation $\lambda = e^{k\lambda}$ and

$$B = \frac{1-k-\sqrt{(1-k)^2-4A}}{2}$$

where A is given by (5). *Then all solutions of Eq. (1) oscillate.* ■

Proof. Assume, for the sake of contradiction, that $x(t)$ is an eventually positive solution of Eq. (1). Then, by Lemma 3, we obtain (6) which, in view of Lemma 2, contradicts (11). The proof is complete. ■

Remark 2 *It is clear that in the above Theorem ω can be replaced by ω_0, where ω_0 is given in Remark 1.*

Remark 3 *Observe that when $\omega = 1$, then (11) reduces to*

$$L > \frac{\ln\lambda_1 - 1 + \sqrt{5-2\lambda_1+2k\lambda_1}}{\lambda_1}, \tag{12}$$

since from [12] it follows that

$$B = 1 - k - \frac{1}{\lambda_1}.$$

In the case that $k = \frac{1}{e}$, then $\lambda_1 = e$ and (12) leads to

$$L > \frac{\sqrt{7-2e}}{e} \approx 0.459987065.$$

Example. Consider the delay differential equation

$$x'(t) + px(t - a\sin^2\sqrt{t} - \frac{1}{pe}) = 0, \tag{13}$$

where $p > 0$, $a > 0$ and $pa = 0.46 - \frac{1}{e}$. Then

$$k = \lim_{t\to\infty} \inf \int_{\tau(t)}^{t} pds = \lim_{t\to\infty} \inf\, p(a\sin^2\sqrt{t} + \frac{1}{pe}) = \frac{1}{e}$$

and

$$L = \lim_{t\to\infty} \sup \int_{\tau(t)}^{t} pds = \lim_{t\to\infty} \sup\, p(a\sin^2\sqrt{t} + \frac{1}{pe}) = pa + \frac{1}{e} = 0.46.$$

Thus, according to Remark 2, all solutions of Eq. (13) oscillate. Observe that none of the results mentioned in the introduction apply to this equation. ■

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